



Concrete-Representational-Abstract: Instructional Sequence for Mathematics

Students, including students who receive special education services, should conceptually understand, as well as develop, accurate execution of standard algorithms to solve computational problems in mathematics (National Mathematics Advisory Panel, 2008). This requires a combined instructional focus on procedural learning and conceptual understanding, which is important to ensure all students learn and master mathematics standards, as outlined by the Pennsylvania Academic Standards.

Conceptual knowledge is defined as understanding the fundamental

tenets of a domain, as well as the connections between concepts within a domain (Flores, Hinton, and Strozier, 2014). This type of knowledge is fluid and can be applied across multiple mathematical contexts. Procedural knowledge is defined as the ability to execute a sequence of mathematical manipulations to solve a problem (Flores, Hinton, and Strozier, 2014). To prepare students for success in the mathematics classroom, the concrete-representational-abstract (CRA) sequence of instruction provides the framework for meeting the goals outlined by the National Mathematics Advisory Council and

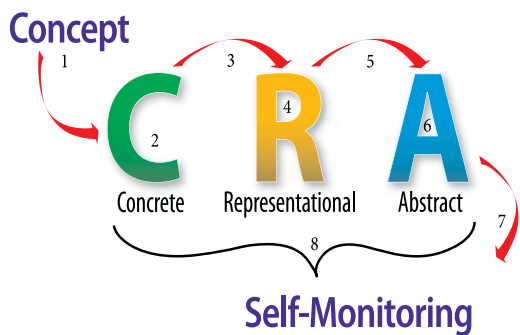
the requirements of the Pennsylvania Academic Standards.

What is CRA?

CRA has its roots in the work of Bruner and Kenney (1965), who defined learning through “Stages of Representation:”

- Enactive, or learning through movement and action
- Iconic, or learning through pictures
- Symbolic, or learning through abstract symbols

The CRA instructional sequence/ approach provides a graduated,



- 1 – Teach prerequisite mathematical concepts.
- 2 – Teach and practice modeling procedures concretely.
- 3 – Connect the concrete to a representation of the concrete.
- 4 – Practice modeling procedures representationally.
- 5 – Connect the representation to the abstract symbols.
- 6 – Practice the abstract modeling of the procedure.
- 7 – Make connections between all three models to help students monitor their thinking and choice of representation.
- 8 – Provide opportunities for student choice.



conceptually supported line of work to create meaningful connections among concrete, representational, and abstract levels of understanding. CRA is an intervention for mathematics instruction that research suggests can enhance the mathematics performance of students in a classroom. It is a three-stage learning process where students learn through physical manipulation of concrete objects, followed by learning through pictorial representations of the concrete manipulations, and ending with solving problems using abstract notation.

Concrete. The “doing” stage uses concrete objects to model problems. In the concrete stage, the teacher begins instruction by modeling each mathematical concept with concrete materials (e.g. red and yellow chips, cubes, base ten blocks, pattern blocks, fraction bars, geometric figures).

Representational. The “seeing” stage uses representations of the objects to model problems. In this stage, the teacher transforms the concrete model into a representational (semiconcrete) level, which may involve drawing pictures; using circles, dots, and tallies; or, using stamps to imprint pictures for counting.

Abstract. The “symbolic” stage uses abstract symbols to model problems. At this stage, the teacher models the mathematical concept at a symbolic level, using numbers, notation, and mathematical symbols to represent the number algorithm. The teacher uses operation symbols (+, ×, ÷, −) to indicate addition, subtraction, multiplication, or division.

How can I use CRA?

CRA may be implemented at all grade levels individually, in small groups, or for the entire class. It can be used with children at the elementary or

secondary level. When using CRA, the teacher should provide multiple opportunities for practice and demonstration to help students achieve mastery of the mathematical concept.

Explicit instruction that involves the use of manipulatives should also include the presentation of the numerical problem (Miller, Stringfellow, Kaffar, Ferreira, & Mancl, 2011).

After multiple teacher demonstrations, students are given opportunities to practice the mathematical model and use verbalization while using the CRA sequence.

What materials do I need to engage in CRA?

Concrete manipulatives such as algebra tiles, cups, sticks, or base ten blocks can be used in the CRA sequence. The teacher should use the manipulative and movements that match the conceptual understanding behind the algorithm.

Why should I use CRA methods?

There is a positive impact for students:

- Interaction with concrete materials enhances student retention of stepwise procedural options in mathematical problem solving. Concrete materials allow students to encode and retrieve information in a variety of sensory options: visual, auditory, tactile, and kinesthetic (Witzel, 2005).
- Students learning secondary level math benefit from engaging with properly designed concrete materials to support the learning of math concepts (Witzel, 2005).
- Under the instruction of a middle school math teacher, results indicated that students who learned with CRA methods to solve algebra transformation equations outperformed peers receiving traditional instruction (Witzel, Mercer, & Miller, 2003).
- CRA provides an opportunity for increased interaction with content and increased frequency of response for all students (Witzel, 2005).
- After receiving CRA instruction, students significantly improved their conceptual understanding of equivalent fractions. While engaging in a post-test, students would fall back on the representational drawings to guide their way through a problem (Butler, Miller, Crehan, Babbitt, & Pierce, 2003).
- In a controlled study, students with disabilities receiving CRA instruction performed as well as general education 8th grade

students on all fractions assessments. They also outperformed general education students on word problem subtests that had fractions and equivalency embedded in the problem (Butler, Miller, Crehan, Babbitt, & Pierce, 2003).

- Students who rely on memorizing procedural steps and lack the conceptual understanding related to foundational operations will not understand why steps are used (Mancl, Miller, & Kennedy, 2012).
- CRA methods to teach place value, fractions, addition, subtraction, multiplication, algebra, and word problems are supported by research (Mancl, Miller, & Kennedy, 2012).
- Early quantitative competencies that children must possess include the relation between number words, numerals and the quantities they represent, fluently manipulating those representations, knowledge of the number line, and basic arithmetic (Geary, 2011).

Students with disabilities struggle to understand the core concepts that underlie operations and algorithms used to solve problems that involve whole numbers and ratios (Butler, Miller, Crehan, Babbitt, & Pierce, 2003). These deficiencies become even more critical with the National Council of Teachers of Mathematics (NCTM) charge to focus on conceptual understanding and problem solving rather than rule-driven knowledge. CRA methods can bridge the conceptual understanding to operational procedure for students.

Examples

Base Ten Blocks for Place Value

Concrete	Representational	Abstract
		342 $300+4+2$

Chips for Equivalent Fractions

Concrete	Representational	Abstract
		$\frac{2}{3} \times 3 = \frac{\square}{9}$

Algebra Tiles for Adding Integers

Concrete	Representational	Abstract
		$-3+5=2$

References:

Butler, F.M., Miller, S.P., Crehan, K., Babbitt, B., and Pierce, T. (2003). *Fraction Instruction for Students With Mathematics Disabilities: Comparing Two Teaching Sequences*. Learning Disabilities Research & Practice. 18(2), 99-111.

Bruner & Kenney. (1965). *Monographs of Society for Research in Child Development*. Wiley, 1-75.

Flores, M.M., Hinton, V., Strozier, S. (2014). *Teaching Subtraction and Multiplication With Regrouping Using the Concrete-Representational-Abstract Sequence and Strategic Instruction Model*. Learning Disabilities Research & Practice. 29(2), 75-88.

Geary, D. C. (2011). *Cognitive predictors of individual differences in achievement growth in mathematics: A five year longitudinal study*. Developmental Psychology, 47, 1539-1552.

National Mathematics Advisory Panel Report (2008). Retrieved from U. S. Department of Education website: <http://www2.ed.gov>

Mancl, D.B., Miller, S.P., & Kennedy, M. (2012). *Using the Concrete –Representational– Abstract Sequence With Integrated Strategy Instruction to Teach Subtraction With Regrouping to Students With Learning Disabilities*. Learning Disabilities Research & Practice. 27(4), 152-166.

Miller, S. P., Stringfellow, B.K., Ferreira, D., Mancl, D.B. (2011). *Developing Computation Competence Among Students Who Struggle With Mathematics*. Teaching Exceptional Children. 44(2), 36-46.

Witzel, B.S. (2005). *Using CRA to Teach Algebra To Students With Math Difficulties in Inclusive Settings*. Learning Disabilities: A Contemporary Journal. 3(2), 49-60.

Witzel, B., Mercer, C. & Miller, D. (2003). *Teaching Algebra to Students With Learning Difficulties: An Investigation of an Explicit Instruction Model*. Learning Disabilities Research & Practice. 18(2), 121-131.

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